Fill Ups of Circle

Q.1. If A and B are points in the plane such that PA/PB = k (constant) for all P on a given circle, then the value of k cannot be equal to .... (1982 - 2 Marks)
Ans. 1
Sol. As P lies on a circle and A and B two points in the plane

such that $\frac{PA}{PB} = k$

Then k can be any real number except 1 as otherwise P will lie on perpendicular bisector of AB which is a line.

Q.2. The points of intersection of the line 4x – 3y – 10 = 0 and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are .... and .... (1983 - 2 Marks)

Ans. (4, 2), (–2, –6)

Sol. For point of intersection of line 4x – 3y – 10 = 0 … (1)

and circle $x^2 + y^2 - 2x + 4y - 20 = 0$ … (2)

Solving (1) and (2), we get

$$\left(\frac{3y+10}{4}\right)^2 + y^2 - 2\left(\frac{3y+10}{4}\right) + 4y - 20 = 0$$

$\Rightarrow y^2 + 4y - 12 = 0 \Rightarrow y = 2, -6$

$\Rightarrow x = 4, -2$

$\therefore$ Points are (4, 2) and (– 2, – 6)

Q.3. The lines 3x – 4y + 4 = 0 and 6x – 8y – 7 = 0 are tangents to the same circle. The radius of this circle is .... (1984 - 2 Marks)

Ans. 3/4

Sol. Let 3x – 4y + 4 = 0 be the tangent at point A and 6x – 8y – 7 = 0 be the tangent of point B of the circle.

As the two tangents parallel to each other

∴ AB should be the diameter of circle.

∴ AB = distance between parallel lines

$3x - 4y + 4 = 0$ and

$6x - 8y - 7 = 0$ = distance between

$6x - 8y + 8 = 0$ and $6x - 8y - 7 = 0$

$$= \left| \frac{8+7}{\sqrt{36+64}} \right| = \frac{15}{10} = \frac{3}{2}$$

∴ radius of circle = $\frac{1}{2}(AB) = \frac{3}{4}$

Q.4. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point (4, 5) with a pair of radii form a quadrilateral of area .... (1985 - 2 Marks)

Ans. 8 sq. units

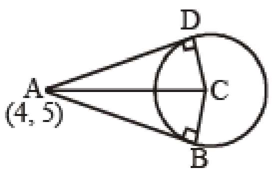
Sol. KEY CONCEPT :

Length of tangent from a point (x_1, y_1) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

The equation of circle is, $x^2 + y^2 - 4x - 2y - 11 = 0$

It's centre is (2, 1), radius = $= \sqrt{4+1+11} = 4 = BC$



length of tangent from the pt. (4, 5) is

$$= \sqrt{16+25-16-10-11} = \sqrt{4} = 2 = AB$$

∴ Area of quad. ABCD

$= 2$ (Area of ΔABC) $= 2 \times \frac{1}{2} \times AB \times BC$

$= 2 \times \frac{1}{2} \times 2 \times 4 = 8$ sq. units.

Q.5. From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the mid-points of these chords is .... (1985 - 2 Marks)

Ans. $x^2 + y^2 - x = 0$

Sol. The equation of given circle is $(x - 1)^2 + y^2 = 1$
or $x^2 + y^2 - 2x = 0$ … (1)

KEY CONCEPT : We know that equation of chord of curve S = 0, whose mid point is (x_1, y_1) is given by $T = S_1$ where T is tangent to curve S = 0 at (x_1, y_1).

∴ If (x_1, y_1) is the mid point of chord of given circle (1), then equation of chord is

$$xx_1 + yy_1 - (x + x_1) = x_1^2 + y_1^2 - 2x_1$$

$$\Rightarrow \quad (x_1 - 1)x + y_1 y + x_1 - x_1^2 - y_1^2 = 0$$

At it passes through origin, we get

$$x_1 - x_1^2 - y_1^2 = 0 \text{ or } x_1^2 + y_1^2 - x_1 = 0$$

$$\therefore \quad \text{locus of } (x_1, y_1) \text{ is } x^2 + y^2 - x = 0$$

Q.6. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is .... (1986 - 2 Marks)

Ans. $10x - 3y - 18 = 0$

Sol. The equation of two circles are

$$x^2 + y^2 - \frac{2}{3}x + 4y - 3 = 0 \text{ ... (1)}$$

and $x^2 + y^2 + 6x + 2y - 15 = 0$ ... (2)

Now we know eq. of common chord of two circles

$S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$

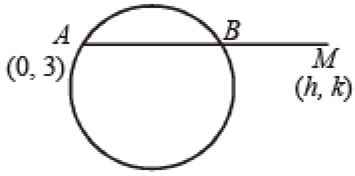
$$\Rightarrow \quad 6x + \frac{2}{3}x + 2y - 4y - 15 + 3 = 0$$

$$\Rightarrow \quad \frac{20x}{3} - 2y - 12 = 0 \Rightarrow 10x - 3y - 18 = 0$$

Q.7. From the point A(0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that AM = 2AB. The equation of the locus of M is .... (1986 - 2 Marks)

Ans.

Sol. The equation of circle is, $x^2 + y^2 + 4x - 6y + 9 = 0$ … (1)



AM = 2AB

⇒ AB = BM

Let the co-ordinates of M be (h, k) Then B is mid pt of AM

$$\therefore \quad B\left(\frac{0+h}{2}, \frac{3+k}{2}\right) = \left(\frac{h}{2}, \frac{k+3}{2}\right)$$

As B lies on circle (1),

$$\therefore \quad \left(\frac{h}{2}\right)^2 + \left(\frac{k+3}{2}\right)^2 + 4 \times \frac{h}{2} - 6\left(\frac{k+3}{2}\right) + 9 = 0$$

$\Rightarrow h^2 + k^2 + 8h - 6k + 9 = 0$

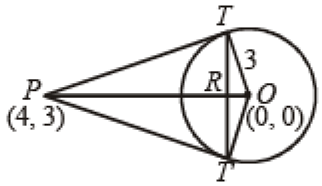
$\therefore$ locus of (h, k) is, $x^2 + y^2 + 8x - 6y + 9 = 0$

Q.8. The area of the triangle formed by the tangents from the point (4, 3) to the the circle $x^2 + y^2 = 9$ and the line joining their points of contact is .... (1987 - 2 Marks)

Ans. 192/25

Sol. From P (4, 3) two tangents PT and PT' are drawn to the circle $x^2 + y^2 = 9$ with O (0, 0) as centre and r = 3.

To find the area of ΔPTT'.



Let R be the point of intersection of OP and TT'.

Then we can prove by simple geometry that OP is perpendicular bisector of TT'.

Equation of chord of contact TT' is 4x + 3y = 9 Now, OR = length of the perpendicular from O to TT' is

$$= \left| \frac{4\times 0+3\times 0-9}{\sqrt{4^2++3^2}} \right| = \frac{9}{5}$$

OT = radius of circle = 3

$$\therefore \quad TR = \sqrt{OT^2 - OR^2} = \sqrt{9-\frac{81}{25}} = \frac{12}{5}$$

Again OP = $\sqrt{(4-0)^2+(3-0)^2} = 5$

$\therefore$ PR = OP - OR = $5-\frac{9}{5}=\frac{16}{5}$

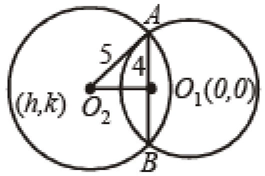
Area of the triangle

PTT' = PR x TR $= \frac{16}{5}\times\frac{12}{5}=\frac{192}{25}$

Q.9. If the circle $C_1 : x^2 + y^2 = 16$ intersects anoth er circle C_2 of radius 5 in such a manner that common chord is of maximum length and has a slope equal to 3/4, then the coordinates of the centre of C_2 are .... (1988 - 2 Marks)

Ans.

Sol. We have $C_1 : x^2 + y^2 = 16$, Centre O_1 (0, 0) radius = 4. C_2 is another circle with radius 5, let its centre O_2 be (h, k).



Now the common chord of circles C_1 and C_2 is of maximum length when chord is diameter of smaller circle C_1, and then it passes through centre O_1 of circle C_1. Given that slope of this chord is 3/4.

∴ Equation of AB is,

$$y = \frac{3}{4}x \Rightarrow 3x - 4y = 0 \;\ldots\; (1)$$

In right ΔAO_1O_2,

$$O_1O_2 = \sqrt{5^2 - 4^2} = 3$$

Also $O_1O_2 = \perp^{lar}$ distance from (h, k) to (1)

$$\Rightarrow \quad 3 = \left|\frac{3h - 4k}{\sqrt{3^2 + 4^2}}\right| \Rightarrow \pm 3 = \frac{3h - 4k}{5}$$

$$\Rightarrow 3h - 4k \pm 15 = 0 \;\ldots\; (2)$$

$$AB \perp O_1O_2 \;\Rightarrow\; m_{AB} \times m_{O_1O_2} = -1$$

$$\Rightarrow \quad \frac{3}{4} \times \frac{k}{h} = -1 \Rightarrow 4h + 3k = 0 \;\ldots\; (3)$$

Solving, 3h – 4k + 15 = 0 and 4h + 3k = 0

We get h = – 9/5, k = 12/5

Again solving 3h – 4k – 15 = 0 and 4h + 3k = 0

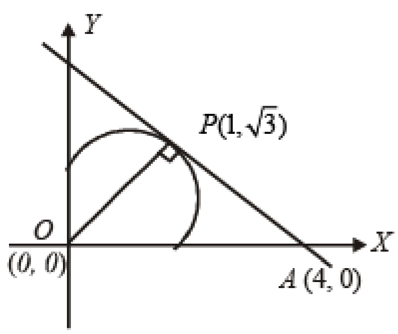
We get h = 9/5, k = –12/5

Thus the required centre is $\left(\frac{-9}{5}, \frac{12}{5}\right)$ or $\left(\frac{9}{5}, \frac{-12}{5}\right)$

Q.10. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is, ... (1989 - 2 Marks)

Ans.

Sol. Tangent at P $(1, \sqrt{3})$ to the circle $x^2 + y^2 = 4$ is $x \cdot 1 + y \cdot \sqrt{3} = 4$



It meets x-axis at A (4, 0),

$\therefore$ OA = 4 Also OP = radius of circle = 2 $\therefore PA = \sqrt{4^2 - 2^2} = \sqrt{12}$

$\therefore$ Area of $\Delta OPA = \frac{1}{2} \times OP \times PA = \frac{1}{2} \times 2 \times \sqrt{12}$

$= 2\sqrt{3}$ sq. units

Q.11. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ = ... (1991 - 2 Marks)

Q.11. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ = ... (1991 - 2 Marks)

Ans. 2

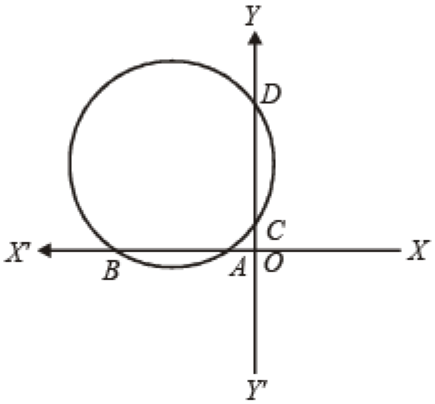
Sol. The given lines are lx – y + 1 = 0 and x – 2y + 3 = 0 which meet x-axis at

$A\left(-\frac{1}{\lambda},0\right)$ and B (– 3, 0)

And

y-axis at C (0, 1) and D $\left(0,\frac{3}{2}\right)$ respectively..

Then we must have, OA × OB = OC × OD



$$\Rightarrow \left(-\frac{1}{\lambda}\right)(-3)=1\times\frac{3}{2} \Rightarrow \lambda=2$$

Q.12. The equation of the locus of the mid-points of the circle $4x^2 + 4y^2 – 12x + 4y + 1 = 0$ that subtend an angle of $2\pi/3$ at its centre is .... (1993 - 2 Marks)

Ans. $16x^2 + 16y^2 – 48x + 16y + 31 = 0$

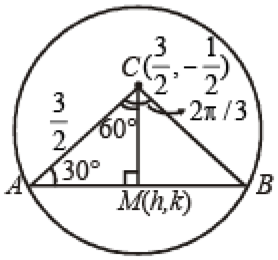
Sol. The given circle is, $4x^2 + 4y^2 – 12x + 4y + 1 = 0$

or $x^2+y^2-3x+y+\frac{1}{4}=0$ with centre $\left(\frac{3}{2},-\frac{1}{2}\right)$

and $r=\sqrt{\frac{9}{4}+\frac{1}{4}-\frac{1}{4}}=\frac{3}{2}$

Let M (h, k) be the mid pt. of the chord AB of the given circle, then CM ⊥ AB. ∠ ACB = 120°.

In ΔACM,



$\angle ACM = \frac{1}{2}\angle ACB = 60°$

and $\angle A = 30°$

$\therefore \quad \sin A = \frac{CM}{AC}$

$\sin 30° = \frac{\sqrt{(h-3/2)^2 + (k+1/2)^2}}{3/2}$

$\Rightarrow \left(\frac{3}{4}\right)^2 = \left(h - \frac{3}{2}\right)^2 + \left(k + \frac{1}{2}\right)^2$

$\Rightarrow 16h^2 + 16k^2 - 48h + 16k + 31 = 0$

$\therefore$ locus of (h, k) is $16x^2 + 16y^2 - 48x + 16y + 31 = 0$

Q.13. The intercept on the line y = x by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is .... (1996 - 1 Mark)

Ans. $x^2 + y^2 - x - y = 0$

Sol. Equation of any circle passin g through the point of intersection of $x^2 + y^2 - 2x = 0$ and y

= x is $x^2 + y^2 - 2x + l\,(y - x) = 0$

or $x^2 + y^2 - (2 + l)x + ly = 0$

Its centre is $\left(\frac{2+\lambda}{2}, \frac{-\lambda}{2}\right)$

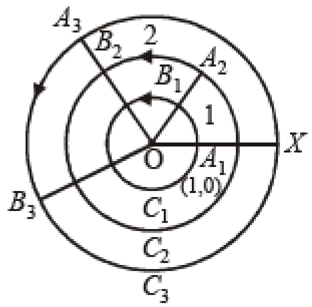
For AB to be the diameter of the required circle, the centre must lie on AB. That is,

$\frac{2+\lambda}{2} = -\frac{\lambda}{2} \Rightarrow \lambda = -1$

Thus, equation of required circle is $x^2 + y^2 - 2x - y + x = 0$ or $x^2 + y^2 - x - y = 0$

Q.14. For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a-particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0).
If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n = ... (1997 - 2 Marks)

Ans. 7



The radius of circle C_1 is 1 cm, C_2 is 2 cm and soon.

It starts from A_1 (1, 0) on C_1, moves a distance of 1 cm on C_1 to come to B_1. The angle subtended by A_1B_1 at the centre

will be $\frac{1}{r} = \theta$ radians, i.e. 1 radian.

From B1 it moves along radius, OB_1 and comes to A_2 on circle C_2 of radius 2. From A_2 it moves on C_2 a distance 2 cm and comes to B_2. The angle subtended by A_2B_2 is again as before 1 radian. The total angle subtended at the centre is 2 radians. The process continues. In order to cross the x-axis again it must describe 2p radians i.e $2.\frac{22}{7} = 6.7$ radians.

Hence it must be moving on circle C_7

$\therefore$ n = 7

Q.15. The chords of contact of the pair of tangents drawn from each point on the line 2x+y =4 to circle x2+y2 = 1 pass through the point .... (1997 - 2 Marks)

Ans.

Sol. Let (h, k) be any point on the given line

$\therefore$ 2h + k = 4 and chord of contact is hx + ky = 1 or hx + (4 – 2h) y = 1

or (4y – 1) + h (x – 2y) = 0

P + l Q = 0.It passes through the intersection of P = 0 and

$Q = 0$ i.e. $\left(\frac{1}{2}, \frac{1}{4}\right)$.

True False of Circle

Q.1. No tangent can be drawn from the point (5/2, 1) to the circumcircle of the triangle with vertices $(1, \sqrt{3})$ $(1, -\sqrt{3})$, $(3, -\sqrt{3})$ (1985 - 1 Mark)

Ans. T

Sol. The circle passes through the points $A(1, \sqrt{3}), B(1, -\sqrt{3})$ and $C(3, -\sqrt{3})$
Here line AB is parallel to y-axis and BC is parallel to x-axis, there $\angle ABC = 90°$

∴ AC is a diameter of circle.

∴ Eq. of circle is

$(x - 1)(x - 3) + (y - \sqrt{3})(y + \sqrt{3}) = 0$
$\Rightarrow x2 + y2 - 4x = 0$ ... (1)
Let us check the position of pt (5/2, 1) with. respect to the circle (1),
we get $S_1 = \frac{25}{4} + 1 - 10 < 0$

∴ Point lies inside the circle.

∴ No tangent can be drawn to the given circle from point (5/2, 1).

∴ Given statement is true.

Q.2. The line x + 3y = 0 is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. (1989 - 1 Mark)

Ans. T

Sol. The centre of the circle $x^2 + y^2 - 6x + 2y = 0$ is (3, – 1)
which lies on the line x + 3y = 0

∴ The statement is true.

Subjective questions of Circle

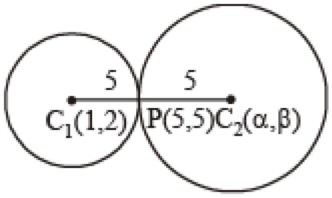
Q.1. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5). (1978)

Ans. $x^2 + y^2 - 18x - 16y + 120 = 0$

Sol. The given circle is

$x^2 + y^2 - 2x - 4y - 20 = 0$ whose centre is (1, 2) and radius = 5 Radius of required circle is also 5.

Let its centre be C_2 (α, β). Both the circles touch each other at P (5, 5).



It is clear from figure that P (5, 5) is the mid-point of C_1C_2.

Therefore, we should have

$\frac{1+\alpha}{2} = 5$ and $\frac{2+\beta}{2} = 5 \Rightarrow \alpha = 9$ and $\beta = 8$

∴ Centre of requir ed cir cle is (9, 8) and equation of required circle is

$(x - 9)^2 + (y - 8)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0$

Q.2. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$.

Suppose that the tangents at the points B(1, 7) and D(4. –2) on the circle meet at the point C. Find the area of the quadrilateral ABCD. (1981 - 4 Marks)

Ans. 75 sq. units

Sol. The eq. of circle is

$x^2 + y^2 - 2x - 4y - 20 = 0$

Centre (1, 2), radius = $\sqrt{1+4+20} = 5$

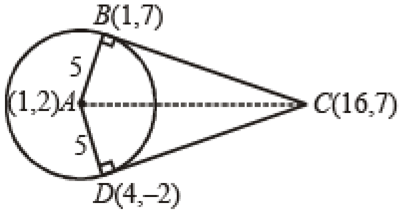
Using eq. of tangent at (x_1, y_1) of

$x^2 + y^2 + 2gx_1 + 2fy_1 + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) f(y + y_1) + c = 0$

Eq. of tangent at (1, 7) is $x \cdot 1 + y \cdot 7 - (x + 1) - 2(y + 7) - 20 = 0$

$\Rightarrow y - 7 = 0$ … (1) Similarly eq. of tangent at (4, – 2) is



$4x - 2y - (x + 4) - 2(y - 2) - 20 = 0$

$\Rightarrow 3x - 4y - 20 = 0$ … (2)

For pt C, solving (1) and (2),

we get x = 16, y = 7 $\therefore$ C (16, 7).

Now, clearly ar (quad ABCD) = 2 Ar (rt ΔABC)

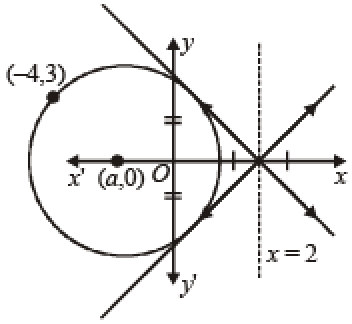
$= 2 \times \frac{1}{2} \times AB \times BC = AB \times BC$

where AB = radius of circle = 5 and BC = length of tangent from C to circle

$= \sqrt{16^2 + 7^2 - 32 - 28 - 20} = \sqrt{225} = 15$

$\therefore$ ar (quad ABCD) = 5 × 15 = 75 sq. units.

Q.3. Find the equations of the circle passing through (–4, 3) and touching the lines x + y = 2 and x – y = 2. (1982 - 3 Marks)



As centre lies on ∠ bisector of given equations (lines) which are the lines y = 0 and x = 2.

∴ Centre lies on x axis or x = 2.

But as it passes through (– 4, 3), i.e., II quadrant.

∴ Centre must lie on x-axis Let it be (a, 0) then distance between (a, 0) and (– 4, 3) is = length of ⊥ lardistance from (a, 0) to x + y – 2 = 0

$$\Rightarrow (a+4)^2 + (0-3)^2 = \left(\frac{a-2}{\sqrt{2}}\right)^2$$

$$\Rightarrow a^2 + 20a + 46 = 0 \Rightarrow a = -10 \pm \sqrt{54}$$

∴ Equation of circle is

$$\Rightarrow [x + (10 \pm \sqrt{54})]^2 + y^2 = [-(10 \pm \sqrt{54}) + 4]^2 + 3^2$$

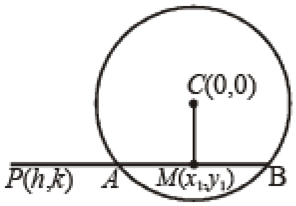
$$\Rightarrow x^2 + y^2 + 2(10 \pm \sqrt{54})x + 8(10 \pm \sqrt{54}) - 25 = 0$$

$$\Rightarrow x^2 + y^2 + 2(10 \pm \sqrt{54})x + 55 \pm \sqrt{54} = 0.$$

Q.4. Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the mid-points of the secants intercepted by the circle is $x^2 + y^2 = hx + ky$. (1983 - 5 Marks)

Ans. Sol. Equation of chord whose mid point is given is

$T = S_1$



[Consider (x_1, y_1) be mid pt. of AB]

$\Rightarrow xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2$

As it passes through (h, k),

$\therefore hx_1 + ky_1 = x_1^2 + y_1^2$

$\therefore$ locus of (x_1, y_1) is, $x^2 + y^2 = hx + ky$

Q.5. The abscissa of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter. (1984 - 4 Marks)

Ans.

Sol. Let the two points be $A = (\alpha_1, \beta_1)$ and $B = (\alpha_2, \beta_2)$

Thus α_1, α_2 are roots of $x^2 + 2\alpha x - b^2 = 0$

$\therefore\ \alpha_1 + \alpha_2 = -2\alpha \ldots (1)$

$\alpha_1 \alpha_2 = -\beta_2 \ldots (2)$

β_1, β_2 are roots of $x^2 + 2px - q^2 = 0$

$\therefore\ \beta_1 + \beta_2 = -2p \ldots (3)$

$\beta_1\beta_2 = -q2 \ldots (4)$

Now equation of circle with AB as diameter is $(x - \alpha_1)(x - \alpha_2) + (y - \beta_1)(y - \beta_2) = 0$

$\Rightarrow x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2 + y^2 - (\beta_1 + \beta_2)y + \beta_1\beta_2 = 0$

$\Rightarrow x^2 + 2\alpha x - \beta_2 + y^2 + 2py - q^2 = 0$ [Using eq. (1), (2), (3) and (4)]

$\Rightarrow x^2 + y^2 + 2\alpha x + 2py - b^2 - q^2 = 0$

Which is the equation of required circle, with its centre (– a, – p) and radius $\sqrt{a^2 + p^2 + b^2 + q^2}$

Q.6. Lines 5x + 12y – 10 = 0 and 5x – 12y – 40 = 0 touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. (1986 - 5 Marks)

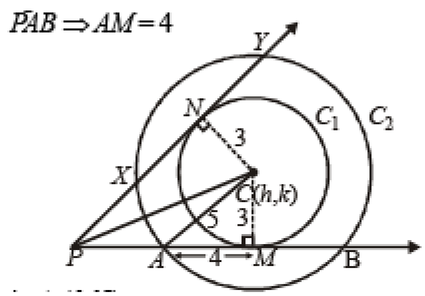
Ans. Sol. Let equation of tangent PAB be 5x + 12y – 10 = 0 and that of

PXY be 5x – 12y – 40 = 0

Now let centre of circles C_1 and C_2 be C (h, k).

Let CM ⊥ PAB then CM = radius of C_1 = 3

Also C_2 makes an intercept of length 8 units on PAB ⇒ AM = 4



Then in ΔAMC, we get

$AC = \sqrt{4^2 + 3^2} = 5$

∴ Radius of C_2 is = 5 units

Also, as 5x + 12y – 10 = 0 ... (1)

and 5x – 12y – 40 = 0 ... (2)

are tangents to C_1, length of perpendicular from C to AB = 3 units

$\therefore$ We get $\frac{5h+12k-10}{13}=\pm 3$

$\Rightarrow 5h + 12k - 49 = 0$ ... (i)

or $5h + 12k + 29 = 0$ ... (ii)

Similarly, $\frac{5h-12k-40}{13}=\pm 3$

$\Rightarrow 5h - 12k - 79 = 0$ … (iii)

or $5h - 12k - 1 = 0$ … (iv)

As C lies in first quadrant

$\therefore$ h, k are + ve

$\therefore$ Eq. (ii) is not possible.

Solving (i) and (iii), we get h = 64/5, k = – 5/4

This is also not possible.

Now solving (i) and (iv), we get h = 5, k = 2.

Thus centre for C_2 is (5, 2) and radius 5.

Hence, equation of C_2 is $(x-5)^2 + (y-2)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 10x - 4y + 4 = 0$

Q.7. Let a given line L_1 intersects the x and y axes at P and Q, respectively. Let another line L_2, perpendicular to L1, cut the x and y axes at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. (1987 - 3 Marks)

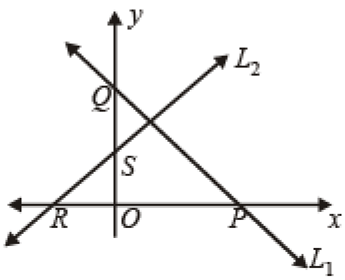
Ans.

Sol. Let the equation of L_1 be $x \cos\alpha + y \sin a = p_1$

Then any line perpendicular to L_1 is $x \sin a - y \cos\alpha = p_2$, where p_2 is a variable.

Then L_1 meets x-axis at P (p_1 secα, 0) and y-axis at Q (0, p_1 cosecα).

Similarly L_2 meets x-axis at R (p_2 cosecα, 0) and y-axis at S (0, – p_2 secα).



Now equation of PS is,

$$\frac{x}{p_1 \sec\alpha} + \frac{y}{-p_2 \sec\alpha} = 1 \quad \Rightarrow \quad \frac{x}{p_1} - \frac{y}{p_2} = \sec\alpha \quad \dots (1)$$

Similarly, equation of QR is,

$$\Rightarrow \quad \frac{x}{p_2 \operatorname{cosec}\alpha} + \frac{y}{p_1 \operatorname{cosec}\alpha} = 1$$

$$\Rightarrow \quad \frac{x}{p_2} + \frac{y}{p_1} = \operatorname{cosec}\alpha \quad \dots (2)$$

Locus of point of intersection of PS and QR can be obtained by eliminating the variable p_2 from (1) and (2)

i.e. $\left(\frac{x}{p_1} - \sec\alpha\right)\frac{x}{y} + \frac{y}{p_1} = \operatorname{cosec}\alpha$

[Substituting the value of $\frac{1}{p_2}$ from (1) in (2)]

$\Rightarrow (x - p_1 \sec\alpha)\, x + y^2 = p_1 y \operatorname{cosec}\alpha$

$\Rightarrow x^2 + y^2 - p_1 x \sec\alpha - p_1 y \operatorname{cosec}\alpha = 0$

which is a circle through origin.

Q.8. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k(x^2 + y^2)^{1/2} = 0$. Find k. (1987 - 4 Marks)

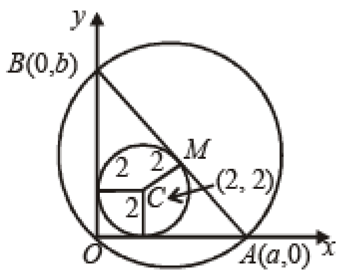
Ans. Sol. The given circle is $x^2 + y^2 - 4x - 4y + 4 = 0$.

This can be re-written as $(x - 2)^2 + (y - 2)^2 = 4$

which has centre C (2, 2) and radius 2.

Let the eq. of third side AB of ΔOAB is $\frac{x}{a}+\frac{y}{b}=1$ such that

A (a, 0) and B (0, b)



Length of perpendicular form (2, 2) on AB = radius = CM = 2

$$\therefore \quad \frac{\left|\frac{2}{a}+\frac{2}{b}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=2$$

Since (2, 2) and origin lie on same side of AB

$$\therefore \quad \frac{-\left(\frac{2}{a}+\frac{2}{b}-1\right)}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=2$$

$$\Rightarrow \quad \frac{2}{a}+\frac{2}{b}-1=-2\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} \quad \dots (1)$$

Since $\angle AOB = \pi/2$.

Hence, AB is the diameter of the circle passing through ΔOAB, mid point of AB is the centre of the circle i.e. (a/2, b/2)

Let centre be (h, k) $\equiv\left(\frac{a}{2},\frac{b}{2}\right)$

then a = 2h, b = 2k.

Substituting the values of a and b in (1), we get

$$\frac{2}{2h}+\frac{2}{2k}-1=-2\sqrt{\frac{1}{4h^2}+\frac{1}{4k^2}}$$

$$\Rightarrow \quad \frac{1}{h}+\frac{1}{k}-1=-\sqrt{\frac{1}{h^2}+\frac{1}{k^2}} \Rightarrow h+k-hk+\sqrt{h^2+k^2}=0$$

∴ Locus of M (h, k) is,

$$x+y-xy+\sqrt{x^2+y^2}=0 \ldots (2)$$

Comparing it with given equation of locus of circumcentre of Δ i.e.

$$x+y-xy+k\sqrt{x^2+y^2}=0 \ldots (3)$$

We get, k = 1

Q. 9. If $\left(m_i,\frac{1}{m_i}\right), m_i>0,$ **i = 1, 2, 3, 4 are four distinct points on a circle, then show that** $m_1m_2m_3m_4 = 1$ **(1989 - 2 Marks)**

Ans. Sol. Given that $\left(m_i,\frac{1}{m_i}\right), m_i>0$, i = 1, 2, 3, 4 are four distinct points on a circle.

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

As the point $\left(m,\frac{1}{m}\right)$ lies on it, therefore, we have

$$m^2+\frac{1}{m^2}+2gm+\frac{2f}{m}+c=0$$

$\Rightarrow m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

Since m_1, m_2, m_3, m_4 are roots of this equation, therefore product of roots = 1

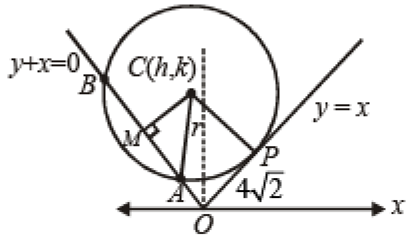
$\Rightarrow m_1m_2m_3m_4 = 1$

Q.10. A circle touches the line y = x at a point P such th at $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (– 10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Determine the equation of the circle. (1990 - 5 Marks)

Ans. $x^2 + y^2 + 18x - 2y + 32 = 0$

Sol. Let AB be the length of ch ord intercepted by circle on y + x = 0

Let CM be perpendicular to AB from centre C (h, k).



Also y – x = 0 and y + x = 0 are perpendicular to each other.

∴ OPCM is rectangle.

∴ CM = OP = $4\sqrt{2}$

Let r be the radius of cirlce.

$$\text{Also } AM = \frac{1}{2}AB = \frac{1}{2} \times 6\sqrt{2} = 3\sqrt{2}$$

∴ In ΔCAM, $AC^2 = AM^2 + MC^2$

$$\Rightarrow \quad r^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2 \Rightarrow r^2 = (5\sqrt{2})^2$$

$$\Rightarrow \quad r = 5\sqrt{2}$$

Again y = x is tangent to the circle at P

∴ CP = r

$$\Rightarrow \quad \left|\frac{h-k}{\sqrt{2}}\right| = 5\sqrt{2} \Rightarrow h-k = \pm 10 \quad (1)$$

Also CM = $4\sqrt{2}$

$$\Rightarrow \left|\frac{h+k}{\sqrt{2}}\right| = 4\sqrt{2} \Rightarrow h+k = \pm 8 \quad \dots (2)$$

Solving four sets of eq's given by (1) and (2), we get the possible centres as (9, – 1), (1, – 9), (– 1, 9), (– 9, 1)

∴ Possible circles are $(x - 9)^2 + (y + 1)^2 - 50$

$= 0 \ (x - 1)^2 + (y + 9)^2 - 50$

$= 0 \ (x + 1)^2 + (y - 9)^2 - 50$

$= 0 \ (x + 9)^2 + (y - 1)^2 - 50 = 0$

But the pt (– 10, 2) lies inside the circle.

∴ $S_1 < 0$

which is satisfied only for $(x + 9)^2 + (y - 1)^2 - 50 = 0$

∴ The required eq. of circle is $x^2 + y^2 + 18x - 2y + 32 = 0$

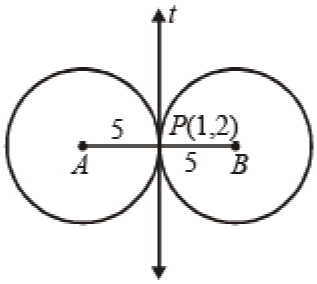
Q.11. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is 4x + 3y = 10, find the equation of the circles. (1991 - 4 Marks)

Ans. Sol. Let t be the common tangent given by $4x + 3y = 10$ … (1)

Common pt of contact being P (1, 2)

Let A and B be the centres of the circles, required.

Clearly, AB is the line perpendicular to t and passing through P (1, 2).



Therefore eq. of AB is

$$\frac{x-1}{4/5}=\frac{x-2}{3/5}=r\left[\begin{array}{l}\text{As slope of } t \text{ is} = -4/3\\ \therefore \text{ slope of } AB \text{ is} = 3/4 = \tan\theta\\ \therefore \cos\theta = 4/5; \sin\theta = 3/5\end{array}\right.$$

For pt A, r = – 5 and for pt B, r = 5, we get

$$\frac{x-1}{4/5}=\frac{y-2}{3/5}=-5,5\left(\begin{array}{l}\text{radius of each circle}\\ \text{being } 5, AP = PB = 5\end{array}\right)$$

⇒ For pt A x = – 4 + 1, y = – 3 + 2

and For pt B x = 4 + 1, y = 3 + 2

∴ A (– 3, – 1) B (5, 5).

∴ Eq.'s of required circles are $(x + 3)^2 + (y + 1)^2 = 5^2$

and $(x – 5)^2 + (y – 5)^2 = 5^2$

$$\Rightarrow \left.\begin{array}{r}x^2+y^2+6x+2y-15=0\\ \text{and } x^2+y^2-10x-10y+25=0\end{array}\right\}$$

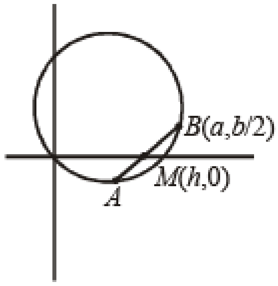
Q.12. Let a circle be given by 2x(x – a) + y(2y – b) = 0, (a ≠ 0, b ≠ 0). Find the condition on a and b if two chords, each bisected by the x- axis, can be drawn to the circle from $\left(a,\frac{b}{2}\right)$ (1992 - 6 Marks)

Ans. Sol. The given circle is 2x (x – a) + y (2y – b) = 0 (a, b ≠ 0)

⇒ $2x^2 + 2y^2 – 2ax – by = 0$ ....(1)

Let us consider the chord of this circle which passes through

the pt $\left(a,\frac{b}{2}\right)$ and whose mid pt. lies on x-axis.



Let (h, 0) be the mid point of the chord, then eq. of chord can be obtained by $T = S_1$

i.e., $2xh + 2y.0 - a(x + h) - \frac{b}{2}(y + 0) = 2h^2 - 2ah$

$\Rightarrow (2h - a)x - \frac{b}{2}y + ah - 2h^2 = 0$

This chord passes through $\left(a, \frac{b}{2}\right)$ therefore

$(2h - a)a - \frac{b}{2} \cdot \frac{b}{2} + ah - 2h^2 = 0$

$\Rightarrow 8h^2 - 12ah + (4a^2 + b^2) = 0$

As given in question, two such chords are there, so we should have two real and distinct values of h from the above quadratic in h, for which

$D > 0$

$\Rightarrow (12a)^2 - 4 \times 8 \times (4a^2 + b^2) > 0$

$\Rightarrow a^2 > 2b^2$

Q.13. Consider a family of circles passing through two fixed points A (3,7) and B (6, 5). Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinate of this point. (1993 - 5 Marks)

Sol. Let the family of circles, passing through A (3, 7) and B (6, 5), be

$x^2 + y^2 + 2gx + 2fy + c = 0$

As it passes through (3, 7)

$\therefore 9 + 49 + 6g + 14f + c = 0$ or, $6g + 14f + c + 58 = 0$ … (1)

As it passes through (6, 5)

$\therefore 36 + 25 + 12g + 10f + c = 0$

$12g + 10f + c + 61 = 0$ … (2)

(2) – (1) gives,

$6g - 4f + 3 = 0 \Rightarrow g = \frac{4f-3}{6}$

Substituting the value of g in equation (1),

we get 4f – 3 + 14f + c + 58 = 0

$\Rightarrow$ 18f + 55 + c = 0 $\Rightarrow$ c = – 18f – 55

Thus the family is

$$x^2 + y^2 + \left(\frac{4f-3}{3}\right)x + 2fy - (18f + 55) = 0$$

Members of this family are cut by the circle $x^2 + y^2 - 4x - 6y - 3 = 0$

∴ Equation of family of chords of intersection of above two circles is $S_1 - S_2 = 0$

$$\Rightarrow \left(\frac{4f-3}{3} + 4\right)x + (2f + 6)y - 18f + 52) = 0$$

which can be written as

$$(3x + 6y - 52) + f\left(\frac{4}{3}x + 2y - 18\right) = 0$$

which represents the family of lines passing through the pt. of intersection of the lines

3x + 6y – 52 = 0 and 4x + 6y – 54 = 0

Solving which we get x = 2 and y = 23/3.

Thus the required pt. of intersection is $\left(2, \frac{23}{3}\right)$

Q.14. Find the coordinates of the point at which the circles $x^2 + y^2 – 4x – 2y = –4$ and $x^2 + y^2 – 12x – 8y = – 36$ touch each other. Also find equations common tangents touching the circles in the distinct points. (1993 - 5 Marks)

Ans. Sol. The given circles are

$x^2 + y^2 – 4x – 2y = – 4$ and

$x^2 + y^2 – 12x – 8y = – 36$

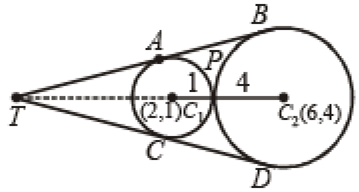
i.e., $x^2 + y^2 – 4x – 2y + 4 = 0$ … (1)

$x^2 + y^2 – 12x – 8y + 36 = 0$ .... (2)

with centres $C_1(2, 1)$ and $C_2 (6, 4)$ and radii 1 and 4 respectively.

Also $C_1C_2 = 5$ As $r_1 + r_2 = C_1C_2$

⇒ Two circles touch each other externally, at P.



Clearly, P divides C_1C_2 in the ratio 1 : 4

∴ Co-ordinates of P are

$$\left(\frac{1\times6+4\times2}{1+4},\frac{1\times4+4\times1}{4+1}\right)=\left(\frac{14}{5},\frac{8}{5}\right)$$

Let AB and CD be two common tangents of given circles, meeting each other at T. Then T divides C_1C_2externally in the ratio 1 : 4.

KEY CONCEPT : [As the direct common tangents of two circles pass through a pt. which divides the line segment joining the centres of two circles externally in the ratio of their radii.]

Hence,, $T\equiv\left(\frac{1\times6-4\times2}{1-4},\frac{1\times4-4\times1}{1-4}\right)=\left(\frac{2}{3},0\right)$

Let m be the slope of the tangent, then equation of tangent through (2/3, 0) is

$$y-0=m\left(x-\frac{2}{3}\right) \Rightarrow y-mx+\frac{2}{3}m=0$$

Now, length of perpendicular from (2, 1), to the above tangent is radius of the circle

$$\therefore \left|\frac{1-2m+\frac{2}{3}m}{\sqrt{m^2+1}}\right|=1$$

$$\Rightarrow (3-4m)^2 = 9(m^2+1) \Rightarrow 9-24m+16m^2 = 9m^2+9$$

$$\Rightarrow 7m^2-24m = 0 \Rightarrow m = 0, \frac{24}{7}$$

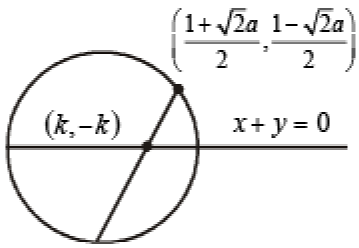
Thus the equations of the tangents are y = 0 and 7y – 24x + 16 = 0.

Q.15. Find the intervals of values of a for which the line y + x = 0 bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2},\frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2+2y^2-(1+\sqrt{2}a)x-(1-\sqrt{2}a)y=0$. (1996 - 5 Marks)

Sol. Let the given point be

$(p,\bar{p})=\left(\frac{1+\sqrt{2}a}{2},\frac{1-\sqrt{2}a}{2}\right)$ and the equation of the circle

becomes $x^2+y^2-px-\bar{p}y=0$



Since the chord is bisected by the line x + y = 0, its mid-point can be chosen as (k, – k). Hence the equation of the chord by $T = S_1$ is

$$kx - ky - \frac{p}{2}(x + k) - \frac{\bar{p}}{2}(y - k) = k^2 + k^2 - pk + \bar{p}k$$

It passes through $A(p, \bar{p})$

$$\therefore \quad kp - k\bar{p} - \frac{p}{2}(p + k) - \frac{\bar{p}}{2}(\bar{p} - k) = 2k^2 - pk + \bar{p}k$$

or $3k(p - \bar{p}) = 4k^2 + (p^2 + \bar{p}^2)$ ... (1)

Put $p - \bar{p} = a\sqrt{2}, p^2 - \bar{p}^2 = 2.\frac{(1+2a^2)}{4} = \frac{1+2a^2}{2}$ ... (2)

Hence, from (1) by the help of (2), we get

$$4k^2 - 3\sqrt{2}ak + \frac{1}{2}(1 + 2a^2) = 0 \text{ ... (3)}$$

Since, there are two chords which are bisected by x + y = 0, we must have two real values of k from (3)

$\therefore \Delta > 0$

or $18a^2 - 8(1 + 2a^2) > 0$

or, $a^2 - 4 > 0$

or, $(a + 2)(a - 2) > 0$

$\therefore a < -2$ or > 2

$\therefore a \in (-\infty, -2) \cup (2, \infty)$

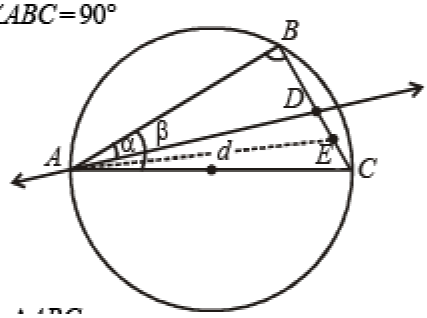
or $a \in]-\infty, -2[\cup]2, \infty[$

Q.16. A circle passes through three points A, B and C with the line segment AC as its diameter. A line passing through A intersects the chord BC at a point D inside the circle. If angles DAB and CAB are α and β respectively and the distance between the point A and the mid point of the line segment DC is d, prove that the area of the circle is

$$\frac{\pi d^2 \cos^2 \alpha}{\cos^2 \alpha + \cos^2 \beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}$$ **(1996 - 5 Marks)**

Sol. Let r be the radius of circle, then AC = 2r Since, AC is the diameter

$\therefore \quad \angle ABC = 90°$



∴ In ΔABC BC = 2r sinβ, AB = 2r cosβ

In rt ∠ed ΔABC BD = AB tanα = 2r cosβ tanα

AD = AB secα = 2r cosβ secα

∴ DC = BC – BD = 2r sinβ – 2r cosβ tanα

Now since E is the mid point of DC

$$\therefore \quad DE = \frac{DC}{2} = \frac{2r\sin\beta - 2r\cos\beta\tan\alpha}{2}$$

⇒ DE = r sinβ – r cosβ tanα Now in ΔADC, AE is the median

$\therefore 2\,(AE^2 + DE^2) = AD^2 + AC^2$

$\Rightarrow 2\,[d^2 + r^2(\sin\beta - \cos\beta\tan\alpha)^2]$

$= 4r^2\cos^2 b\sec^2 a + 4r^2$

$$\Rightarrow \quad r^2 = \frac{d^2\cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

⇒ Area of circle,

$$\pi r^2 = \frac{\pi d^2\cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

Q.17. Let C be any circle with centre $(0, \sqrt{2})$. Prove that at the most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers.) (1997 - 5 Marks)

Sol. Given C is the circle with centre at (0, 2) and radius r (say)
then $C \equiv x^2 + (y - \sqrt{2})^2 = r^2$

$$\Rightarrow (y-\sqrt{2})^2 = (r^2 - x^2) \Rightarrow y - \sqrt{2} = \pm\sqrt{r^2 - x^2}$$

$$\Rightarrow y = \sqrt{2} \pm \sqrt{r^2 - x^2} \quad \ldots (1)$$

The only rational value which y can have is 0.

Suppose the possible value of x for which y is 0 is x_1. Certainly – x_1 will also give the value of y as 0 (from (1)).

Thus, at the most, there are two rational pts which satisfy the eq^n of C.

Q.18. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. (1998 - 8 Marks)

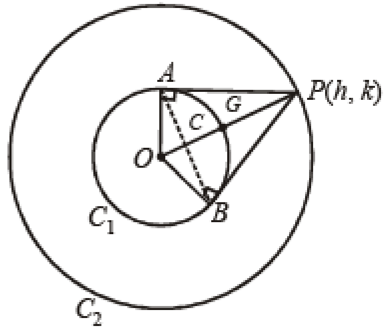
Ans.

Sol. Let P (h, k) be on C_2

$\therefore h^2 + k^2 = 4r^2$

Chord of contact of P w.r.t. C_1 is $hx + ky = r^2$

It intersects C_1, $x^2 + y^2 = a^2$ in A and B.



Eliminating y, we get,

$$x^2+\left(\frac{r^2-hx}{k}\right)^2=r^2$$

or, $x^2(h^2+k^2)-2r^2hx+r^4-r^2k^2=0$

or, $x^2.4r^2-2r^2hx+r^2(r^2-k^2)=0$

$$\therefore \quad x_1+x_2=\frac{2r^2h}{4r^2}=\frac{h}{2}, y_1+y_2=\frac{k}{2}$$

If (x, y) be the centroid of ΔPAB, then

$$3x=x_1+x_2+h=\frac{h}{2}+h=\frac{3h}{2}$$

$\therefore \quad x=\frac{h}{2}$ or h = 2x and similarly k = 2y

Putting in (1) we get $4x^2 + 4y^2 = 4r^2$

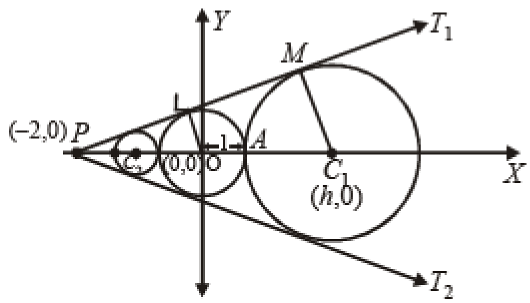
∴ Locus is $x^2 + y^2 = r^2$ i.e., C_1

Q.19. Let T_1, T_2 be two tangents drawn from (–2, 0) onto the circle C : $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time. (1999 - 10 Marks)

Sol. The given circle is $x^2 + y^2 = 1$ … (1)

Centre O (0, 0) radius = 1

Let T_1 and T_2 be the tangents drawn from (– 2, 0) to the circle (1).



Let m be the slope of tangent then equations of tangents are y – 0 = m (x + 2) or, mx – y + 2m = 0 … (2)

As it is tangent to circle (1)

length of ⊥ lar from (0, 0) to (2) = radius of (1)

$$\Rightarrow \left|\frac{2m}{\sqrt{m^2+1}}\right| = 1 \Rightarrow 4m^2 = m^2 + 1 \Rightarrow m = \pm 1/\sqrt{3}$$

∴ Th e two tangen ts are $x + \sqrt{3}y + 2 = 0 (T_1)$ and

$x - \sqrt{3}y + 2 = 0 (T_2)$

Now any other circle touching (1)

and T_1, T_2 is such that its centre lies on x-axis.

Let (h, 0) be the centre of such circle, then from fig.

$OC_1 = OA + AC_1 \Rightarrow |h| = 1 + |AC_1|$

But AC_1 = ⊥ lar distance of (h, 0) to tangent

$$\Rightarrow |h| = 1 + \left|\frac{h+2}{2}\right| \Rightarrow |h| - 1 = \left|\frac{h+2}{2}\right|$$

Squaring,

$$h^2 - 2|h| + 1 = \frac{h^2 + 4h + 4}{4}$$

$\Rightarrow 4h^2 \pm 8h + 4 = h^2 + 4h + 4$

'+' $\Rightarrow 3h^2 = -4h \Rightarrow h = -4/3$

'–' $\Rightarrow 3h^2 = 12h \Rightarrow h = 4$

Thus centres of circles are (4, 0), $\left(-\frac{4}{3}, 0\right)$

∴ Radius of circle with centre (4, 0) is = 4 – 1 = 3 and radius of circle with centre $\left(\frac{-4}{3}, 0\right)$ is $= \frac{4}{3} - 1 = \frac{1}{3}$

∴ The two possible circles are $(x-4)^2 + y^2 = 3^2$ … (3)

$\left(x+\frac{4}{3}\right)+y^2=\left(\frac{1}{3}\right)^2$ … (4)

Now, common tangents of (1) and (3). Since (1) and (3) are two touching circles they have three common tangents

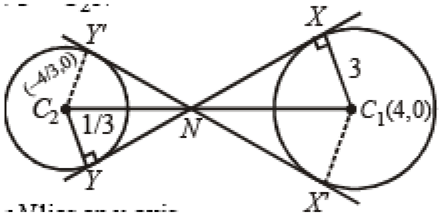
T_1, T_2 and x = 1 (clear from fig.)

Similarly common tangents of (1) and (4) are T_1, T_2 and x = – 1.

For the circles (3) and (4) there will be four common tangents of which two are direct common tangents .

XY and x' y' and two are indirect common tangents. Let us find two common indirect tangents. We know that In two similar Δ's C_1XN and C_2YN

$\frac{3}{1/3}=\frac{C_1N}{C_2N} \Rightarrow N$ divides C_1C_2 in the ratio 9 : 1.



Clearly N lies on x-axis.

$$\therefore \quad N=\left(\frac{9\times(-4/3)+1\times 4}{10},0\right)=\left(\frac{-4}{5},0\right)$$

Any line through N is

$y=m\left(x+\frac{4}{5}\right)$ or 5mx – 5y + 4m = 0

If it is tangent to (3) then

$$\left|\frac{20m+4m}{\sqrt{25m^2+25}}\right|=3$$

$\Rightarrow 24m=15\sqrt{m^2+1} \quad \Rightarrow 64m^2 = 25m^2 + 25$

$\Rightarrow 39m^2 = 25 \Rightarrow m = \pm 5/\sqrt{39}$

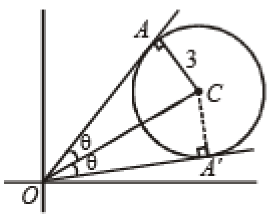
∴ Required tangents are

$$y = \pm\frac{5}{\sqrt{39}}\left(x+\frac{4}{5}\right).$$

Q.20. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. (2001 - 5 Marks)

Ans. Sol. The equation $2x^2 - 3xy + y^2 = 0$ represents pair of tangents OA and OA'. Let angle between these to tangents be 2θ.

Then, $$\tan 2\theta = \frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - 2\times 1}}{2+1}$$



$$\left[\text{Using } \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}\right]$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{3} \Rightarrow \tan^2\theta + 6\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-6\pm\sqrt{36+4}}{2} = -3\pm\sqrt{10}$$

As θ is acute $\tan\theta = \sqrt{10} - 3$

Now we know that line joining the pt through which tangents are drawn to the centre bisects the angle between the tangents,

∴∠AOC = ∠A'OAC = θ

In ΔAOC,

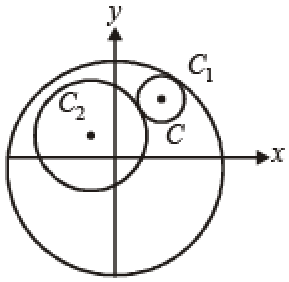
$\tan\theta = \frac{3}{OA} \Rightarrow OA = \frac{3}{\sqrt{10}-3} \times \frac{\sqrt{10}+3}{\sqrt{10}+3}$

$\therefore \quad OA = 3(3+\sqrt{10}).$

Q.21. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally.

Identify the locus of the centre of C. (2001 - 5 Marks)

Ans. Sol. Let equation of C_1 be $x^2 + y^2 = r_1^2$ and of C_2 be $(x - a)^2 + (y - b)^2 = r_2^2$



Let centre of C be (h, k) and radius be r, then by the given conditions.

$\sqrt{(h-a)^2+(k-b)^2} = r + r_2$ and $\sqrt{h^2+k^2} = r_1 - r$

$\Rightarrow \sqrt{(h-a)^2+(k-b)^2} + \sqrt{h^2+k^2} = r_1 + r_2$

Required locus is

$\sqrt{(x-a)^2+(y-b)^2} + \sqrt{x^2+y^2} = r_1 + r_2,$

which represents an ellipse whose foci are at (a, b) and (0, 0).

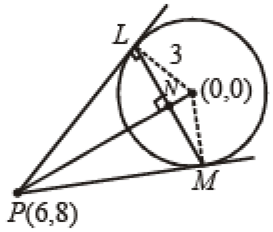
[$\because$ PS + PS' = constant $\Rightarrow$ locus of P is an ellipse with foci at S and S']

Q.22. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point P (6, 8) to the circle and the chord of contact is maximum. (2003 - 2 Marks)

Ans.

Sol. The given circle is $x^2 + y^2 = r^2$

From pt. (6, 8) tangents are drawn to this circle.



Then length of tangent

$$PL = \sqrt{6^2 + 8^2 - r^2} = \sqrt{100 - r^2}$$

Also equation of chord of contact LM is

$6x + 8y - r^2 = 0$

PN = length of ⊥ lar from P to LM

$$= \frac{36 + 64 - r^2}{\sqrt{36 + 64}} = \frac{100 - r^2}{10}$$

Now in rt. ΔPLN, $LN^2 = PL^2 - PN^2$

$$= \frac{(100 - r^2)^2}{100} = \frac{(100 - r^2)r^2}{100} \Rightarrow LN = \frac{r\sqrt{100 - r^2}}{10}$$

$$\therefore \quad LM = \frac{r\sqrt{100 - r^2}}{5} \quad (\because LM = 2\ LN)$$

∴ Area of ΔPLM $= \frac{1}{2}$ x L M x PN

$$= \frac{1}{2} \times \frac{r\sqrt{100 - r^2}}{5} \times \frac{100 - r^2}{10} = \frac{1}{100}[r(100 - r^2)^{\frac{3}{2}}]$$

For max value of area, we should have

$$\frac{dA}{dr} = 0$$

$$\Rightarrow \frac{1}{100}\left[(100 - r^2)^{\frac{3}{2}} + r.\frac{3}{2}(100 - r^2)^{\frac{1}{2}}(-2r)\right] = 0$$

$\Rightarrow \quad (100-r^2)^{\frac{1}{2}}[100-r^2-3r^2]=0$

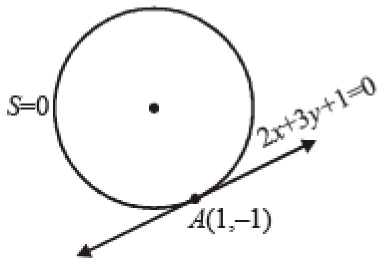
$\Rightarrow r = 10$ or $r = 5$

But $r = 10$ gives length of tangent PL = 0

$\therefore r \neq 10$. Hence, $r = 5$

Q.23. Find the equation of circle touching the line 2x + 3y + 1 = 0 at (1, – 1) and cutting orthogonally the circle having line segment joining (0, 3) and (–2, –1) as diameter. (2004 - 4 Marks)

Ans. Sol. We are given that line $2x + 3y + 1 = 0$ touches a circle $S = 0$ at $(1, -1)$.\



So, eq^n of this circle can be given by $(x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0$.

[Note : $(x-1)^2 + (y+1)^2 = 0$ represents a pt. circle with centre at $(1, -1)$].

or $x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0$ …(1)

But given that this circle is orthogonal to the circle, the extremities of whose diameter are (0, 3) and (– 2, – 1)

i.e. $x(x + 2) + (y - 3)(y + 1) = 0$

$x^2 + y^2 + 2x - 2y - 3 = 0$ .... (2)

Applying the condition of orthogonality for (1) and (2), we

get $2(\lambda-1).1+2\left(\frac{3\lambda+2}{2}\right).(-1)=\lambda+2+(-3)$

$$[2g_1g_2 + 2f_1f_2 = c_1 + c_2]$$

$\Rightarrow 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

$\Rightarrow \quad 2\lambda = -3 \quad \Rightarrow \quad \lambda = \frac{-3}{2}$

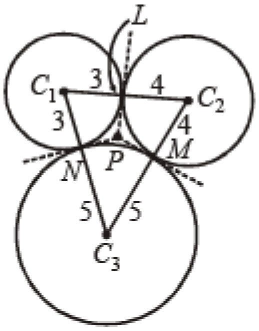
Substituting this value of λ in eq^n (1) we get the required circle as

$$x^2 + y^2 - 5x - \frac{5}{2}y + \frac{1}{2} = 0$$

or, $2x^2 + 2y^2 - 10x - 5y + 1 = 0$

Q.24. Circles with radii 3, 4 and 5 touch each other externally. If P is the point of intersection of tangents to these circles at their points of contact, find the distance of P from the points of contact. (2005 - 2 Marks)

Ans. Sol. Given these circles with centres at C_1, C_2 and C_3 and with radii 3, 4 and 5 respectively, The three circles touch each other externally as shown in the figure.



P is the point of intersection of the three tangents drawn at the pts of contacts, L, M and N. Since lengths of tangents to a circle from a point are equal, we get

PL = PM = PN

Also $PL \perp C_1C_2$, $PM \perp C_2C_3$, $PN \perp C_1C_3$

(Q tangent is perpendicular to the radius at pt. of contact)

Clearly P is the incentre of $\Delta C_1C_2C_3$ and its distance from pt.of contact i.e.,

PL is the radius of incircle of $\Delta C_1C_2C_3$.

In $\Delta C_1C_2C_3$ sides are a = 3 + 4 = 7, b = 4 + 5 = 9, c = 5 + 3 = 8

$PL \perp C_1C_2, PM \perp C_2C_3, PN \perp C_1C_3$

$\therefore \quad s = \dfrac{a+b+c}{2} = 12$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{12 \times 5 \times 3 \times 4} = 12\sqrt{5}$

$\therefore \quad r = \dfrac{\Delta}{s} = \dfrac{12\sqrt{5}}{12} = \sqrt{5}$

Integar Type ques of Circle

Q.1. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segement joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is (2009)

Ans. (8)

Sol. Let r be the radius of required circle.

Clearly, in ΔC_1CC_2 , $C_1C = C_2C = r + 1$

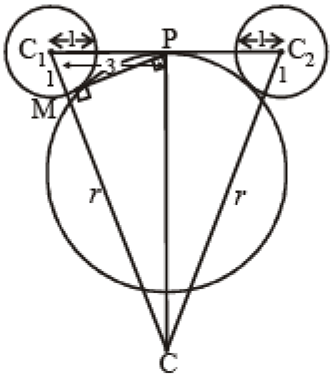
and P is mid point of C_1C_2

$\therefore CP \perp C_1C_2$

Also $PM \perp CC_1$

Now $\Delta PMC_1 \sim \Delta CPC_1$ (by AA similarity)

$$\therefore \frac{MC_1}{PC_1} = \frac{PC_1}{CC_1}$$



$$\Rightarrow \frac{1}{3} = \frac{3}{r+1} \Rightarrow r + 1 = 9 \Rightarrow r = 8.$$

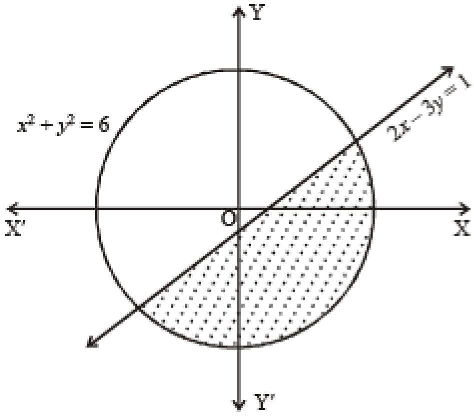
Q.2. The straight line 2x – 3y = 1 divides the circular region $x^2 + y^2 \leq 6$ into two parts.

If $S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right),\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\}$ then the number ofpoints (s) in S lying inside the smaller part is (2011)

Sol.

The smaller region of circle is the region given by $x^2 + y^2 < 6$ ...(1)

and $2x - 3y > 1$ ...(2)



We observe that only two points $\left(2,\frac{3}{4}\right)$ and $\left(\frac{1}{4},-\frac{1}{4}\right)$

satisfy both the inequations (1) and (2)

$\therefore$ 2 points in S lie inside the smaller part.